Efficient Computation of Modal Sensitivities for Systems with Repeated Frequencies

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The physical origin and ambiguity of repeated frequency modes is discussed, followed by a mathematical derivation of the governing equations for the partial derivatives of repeated eigenvalues with respect to changes in the system parameters. This is followed by a derivation for the associated eigenvector derivative equations. Physical interpretations are included to help explain the analytical results. An efficient computation procedure, which preserves the bandwidth for very large systems, is also proposed for systems with repeated or closely spaced eigenvalues. Two small examples to illustrate the solution procedure are presented.

Nomenclature

$[A]$, $[B]$	= $n \times n$ symmetric matrices
b, b_{jk}	= see Eqs. (7), (21), and (25)
C_j, C_{jk}	= see Eqs. (21) and (25)
$[D]$	= see Eq. (18)
$\{F_i\}$	= see Eq. (5)
K, K_1, K_2	= spring constants
M, M_a, M_b	= lumped masses
m	= multiplicity of eigenvalue
r_j	= jth design parameter
T	= kinetic energy or transpose (when used as a superscript)
V	= potential energy
$\{V_i\}$	= solution of Eq. (10)
$\{X\}$, $\{Z\}$	= eigenvectors
$\{\alpha\}$	= see Eq. (12)
α_j	= jth element of $\{\alpha\}$
$\delta(\ )$	= variation of quantity contained in parenthesis
θ, ϕ	= angle variables for two degree-of-freedom example
λ	= eigenvalue
$(\ )'$	= partial derivative with respect to design parameter

Introduction

THE importance of obtaining sensitivities for eigenvalue problems stems from the fact that partial derivatives with respect to system parameters are extremely important for effecting efficient design modifications for given situations, for gaining insight into the reasons for discrepancies between structural analyses and dynamic tests, and for indicating system model changes that will improve correlations between analyses and tests. While the problem of obtaining eigen*vector* sensitivities has been of interest for some time and its efficient solution was presented over a decade ago,[1] cases associated with repeated roots have only recently been addressed.[2,3] However, the eigen*value* sensitivity problem for repeated roots has received attention in recent years by Haug and his collaborators.[4–6]

The situation of repeated frequencies, or identical frequencies with different mode shapes, occurs in many physical situations. Perhaps the most common circumstances under which multiple eigenvalues occur in engineering are instances where system symmetry exists, such as structures with two or more planes of reflective or cyclic symmetry or in the limiting case of axisymmetric bodies. Examples of structures with repeated roots are shown in Figs. 1–3.

Crandall[7] has presented a simple example to explain this phenomenon, in physical terms, through consideration of a frictionless particle sliding back and forth near the bottom of a shallow elliptic bowl (see Fig. 4). He states, "The eigenvalue problem for this system consists in finding the paths and the frequencies of back-and-forth motion for the natural modes of motion in which each back-and-forth excursion of a particle is always on the same path.

"The answer to this problem is simple. The paths of the natural modes are along the major and minor axes of the ellipse. The natural frequencies depend on the radii of curvature of these paths, the lower frequency being associated with the larger radius, i.e., the flattest path. Here the eigenvalues (as measures of the natural frequencies) are different, and with each is associated a unique mode. The orthogonality of the modes shows up in the perpendicularity of the two paths.

"Now imagine that the elliptic bowl is gradually transformed into a circular bowl. The two principal radii of curvature, and hence the eigenvalues, will approach one another. The directions of the paths of the natural modes will remain unchanged in the limiting process; however, in the circular bowl any straight-line path through the bottom of the bowl is equally well the path of a natural mode. Thus, when two eigenvalues coalesce, there is introduced a whole infinity of modes. We may still consider that there are two orthogonal modes with the understanding when the eigenvalues are equal any linear combination of these modes is also a mode."[7]

In addition to the physical problem created by the arbitrariness of the eigenvector for the repeated mode, there is the mathematical problem associated with this phenomenon as well. Mathematically speaking, the arbitrariness is manifested by additional singularities in the associated eigenvalue problem coefficient matrix and the important question as to whether an eigenvalue or eigenvector differential even exists. The authors of Refs. 4–6 have concluded that the eigenvalue derivatives, in the presence of repeated eigenvalues, are only directionally differentiable. We will discuss this briefly later.

It is also possible for closely spaced eigenvalues to occur where physical symmetry is not present, such as in classical flutter when the bending and torsional wing frequencies coalesce. It is easily shown that if two mode shapes possess the same eigenvalue, then any linear combination of these is also a mode shape, since their combination will also satisfy the same linear eigenvalue equation.

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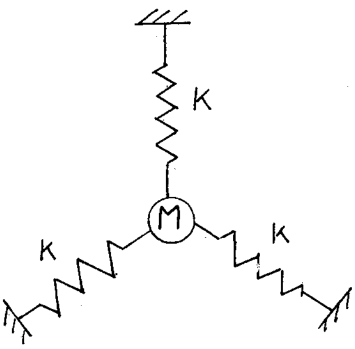

Fig. 1 Two-dimensional vibrations of a symmetrically supported mass.

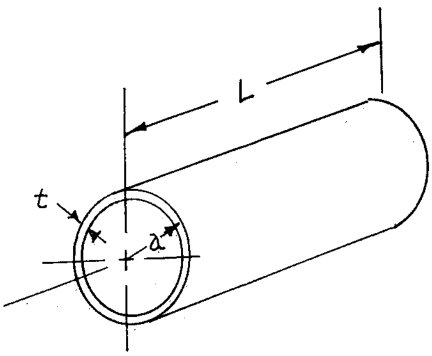

Fig. 2 Vibrations of a right circular cylinder.

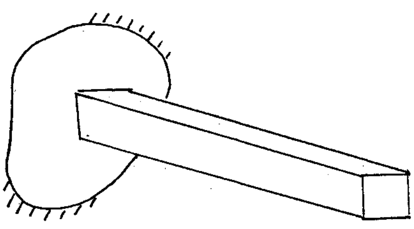

Fig. 3 Vibration of a cantilevered beam.

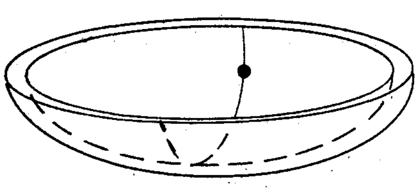

Fig. 4 Vibrations of a frictionless mass in a shallow elliptic dish.

Mathematical Background

As is well known, the $n \times n$ linear eigenvalue equation matrix $([A] - \lambda_i[B])$ is singular since

$$([A] - \lambda_i[B])\{X_i\} = \{0\} \tag{1}$$

In the above equation, we have assumed that $[A]$ and $[B]$ are real symmetric $n \times n$ full or banded matrices. $([A] - \lambda_i[B])$ will be of rank $(n-1)$ if λ_i is not a repeated root, but it will be of rank $n-(m+1)$, where m is the number of times λ_i is repeated. The m repeated eigenvectors, $\{X_{i+j}\}$ ($j = 1,2,\ldots,m$), may be found through the (somewhat artificial) orthogonality conditions

$$\{X_i\}^T[B]\{X_{i+j}\} = 0 \qquad j = 1,2,\ldots,m \tag{2}$$

For simplicity, we shall also assume that all modes are normalized such that

$$\{X_i\}^T[B]\{X_i\} = 1 \tag{3}$$

If we also wish to obtain system derivatives, then a differentiation of Eqs. (1) and (3) yields

$$([A] - \lambda_i[B])\{X_i'\} = \{F_i\} \tag{4}$$

where

$$\{F_i\} = -([A] - \lambda_i[B])'\{X_i\} \tag{5}$$

and

$$\{X_i\}^T[B]\{X_i'\} = b \tag{6}$$

where

$$b = -\tfrac{1}{2}\{X_i\}^T[B']\{X_i\} \tag{7}$$

The gradient of λ_i is given by

$$\lambda_i' = \{X_i\}^T([A]' - \lambda_i[B]')\{X_i\} \tag{8}$$

which is easily obtained from the premultiplication of Eq. (4) by $\{X_i\}^T$ and application of Eqs. (1) and (3).

Partial Derivatives of Eigenvectors

Nonrepeated Roots

If λ_i is not a repeated root, there are numerous ways for obtaining the eigenvalue gradients, some of which involve modal expansions[2,8] and some of which do not.[1,3] Perhaps the most efficient nonmodal expansion methods are due to Nelson[1] and Ojalvo,[3] whose procedures also preserve the bandwidth of the original system matrices. Nelson's procedure involves solving Eq. (8) for λ_i', setting

$$\{X_i'\} = \{V_i\} + C_i\{X_i\} \tag{9}$$

and letting V_i be the solution of

$$(\overline{[A] - \lambda_i[B]})\{V_i\} = \{\bar{F}_i\} \tag{10}$$

where $(\overline{[A] - \lambda_i[B]})$ is equal to $([A] - \lambda_i[B])$ with the pth row and column set to zero, except for the pth element. In the previous discussion, p corresponds to the maximum element of $\{X_i\}$ and $\{\bar{F}_i\}$ is equal to $\{F_i\}$ with the pth element set to zero. The effect of the above modification to $([A] - \lambda_i[B])$ and $\{F_i\}$ is to make the pth element of $\{V_i\}$ equal to zero. Substitution of Eq. (9) into Eq. (6) results in the solution of C_i from the equation

$$C_i = b - \{X_i\}^T[B]\{V_i\} \tag{11}$$

Repeated Roots

Nelson's method will not work if λ_i is a repeated root, since $(\overline{[A] - \lambda_i[B]})$ will still be singular and so Eq. (10) will not yield a unique solution for $\{V_i\}$.

As noted earlier, if λ_i is a repeated eigenvalue, with corresponding eigenvectors $\{X_j\}$, ($j = i, i+1,\ldots,i+m$) then a linear combination of $\{X_j\}$ will also be an eigenvector, i.e.,

$$\{Z_i\} = \sum_{i}^{i+m} \alpha_j\{X_j\} = [X]\{\alpha\} \tag{12}$$

where

$$[X] = \begin{bmatrix} | & | & & | \\ X_i & X_{i+1} & \cdots & X_{i+m} \\ | & | & & | \end{bmatrix} \tag{13}$$

and

$$\{\alpha\} = \begin{Bmatrix} \alpha_i \\ \vdots \\ \alpha_{i+m} \end{Bmatrix}$$

It should be noted that the α_j parameters are completely arbitrary, except for the constraint $\{\alpha\}^T\{\alpha\}=1$ obtained from Eq. (3). In addition, we note that since the vectors $\{X_j\}$ $(j=i, i+1, \ldots, i+m)$ for a repeated eigenvalue are not unique, they may not be continuous and differentiable in r. Therefore, we have postulated that $\{Z_i\}$ [see Eq. (12)] is the correct vector which is continuous and differentiable in r. However, since the correct vector $\{Z_i\}$ is not known a priori, we have left its specificity open initially and will determine the coefficients α_j or vector $\{\alpha\}$ later.

Returning to Crandall's physical model of the spherical dish,[7] we assume that we have selected an arbitrary pair of orthonormal eigenpaths, $\{X_1\}$ and $\{X_2\}$, and that we shall perturb a particular radius by a slight elongation δ_r in an orientation whose direction cosines relative to the direction of $\{X_1\}$ and $\{X_2\}$ are α_j and $(1-\alpha_j)^{1/2}$, respectively. Thus,

$$\{Z_1\} = \alpha_1\{X_1\} + (1-\alpha_1)^{\frac{1}{2}}\{X_2\}$$

The corresponding eigenvector gradient equation for Z_i become

$$([A] - \lambda_i[B])\{Z_i'\} = \{F_i(\lambda_i, Z_i)\} \tag{14}$$

and

$$\{F_i(\lambda_i, Z_i)\} = -([A] - \lambda_i[B])'\{Z_i\} \tag{15}$$

Substitution of Eq. (12) into Eq. (14) and premultiplication by $[X]^T$ yields the $(m+1)$ equations

$$[X]^T([A] - \lambda_i[B])[X]\{\alpha\} = [X]^T\{F_i\} = \{0\} \tag{16}$$

where use has been made of the transpose of Eq. (1) to establish the $m \times 1$ null vector in Eq. (16).

Combination of Eqs. (2), (3), (12), (14), and (16) then yields the auxiliary eigenvalue problem for the eigenvalue gradients, i.e.,

$$[D]\{\alpha\} = \lambda_i'\{\alpha\} \tag{17}$$

where $[D]$ is the $(m+1) \times (m+1)$ matrix given by

$$[D] = [X]^T([A']) - \lambda_i[B'])[X] \tag{18}$$

It is interesting to note that Eq. (17) possesses $m+1$ eigenvalues λ_i', which is to say that λ_i has $m+1$ partial derivatives. This may be physically explained by reference to Fig. 5, which shows how two eigenvalues λ_i and λ_{i+1} depend upon a system parameter r. At the particular value of r where λ_i and λ_{i+1} coalesce, there are obviously two values of λ_i'; if $m+1$ values of λ_i were to coalesce, there would be $m+1$ derivatives. If λ_i' is itself a repeated eigenvalue derivative, then the $\{\alpha\}$ of Eq. (17) are not uniquely determined. Such cases are not addressed in this paper.

Solution of Eq. (17) will also yield $m+1$ values of $\{\alpha\}$, which then uniquely determine the $m+1$ eigenvectors Z_{i+j} corresponding to λ_i.

Thus, it may be seen that selection of the parameters r, considered to be varied for derivative purposes, uniquely defines the original eigenvectors $\{Z_i\}$ once λ_i and $\{\alpha\}$ are determined. To compute $\{Z_i'\}$ from Eq. (14), however, we must first eliminate the $m+1$ redundant equations and replace them with $m+1$ independent equations. This is done by extending Nelson's method[1] through the elimination of those equations, $p, q, s, \ldots$, etc., that correspond to the maximum elements of the $m+1$ eigenvectors $\{Z_i\}$. If any of these repeat (e.g., $p=q$, s, or t), simply select p to correspond to the next largest element of that particular eigenvector, such that no repetitions occur and the equations eliminated correspond to the number of repeated eigenvalues. Once again, perform the equation elimination by zeroing out the corresponding rows and columns of $([A] - \lambda_i[B])$, except for the diagonals, to form $(\overline{[A] - \lambda_i[B]})$, set the corresponding elements of $\{F_i\}$ to form $\{\bar{F}_i\}$ equal to zero, and let

$$\{Z_j'\} = \{V_j\} + \sum_{k=i}^{i+m} C_{jk}\{Z_k\} \qquad j = i, i+1, \ldots, i+m \tag{19}$$

Then, the solution of

$$(\overline{[A] - \lambda_i[B]})\{V_j\} = \{\bar{F}_j\} \qquad j = i, i+1, \ldots, i+m \tag{20}$$

and substitution into Eq. (6) will yield

$$C_{jj} = b_{jj} - \{Z_j\}^T[B]\{V_j\} \tag{21a}$$

with

$$b_{jj} = -\tfrac{1}{2}\{Z\}^T[B']\{Z_j\} \tag{21b}$$

We shall see that the remaining C_{jk} are partially determined from the derivatives of the artificial (but not inconsistent) orthogonality conditions [see. Eq. (2)],

$$\{Z_j\}^T[B]\{Z_k'\} + \{Z_k\}^T[B]\{Z_j'\} = -\{Z_k\}^T[B']\{Z_j\},\ k \neq j \tag{22}$$

It will be seen that Eq. (22) will not produce a sufficient number of equations to uniquely determine all of the C_{jk}. Therefore, we suggest that the following convenient (but arbitrary) symmetrical equations be used to define all the C:

$$\{Z_j\}^T[B]\{Z_k'\} = \{Z_k\}^T[B]\{Z_j'\} \tag{23}$$

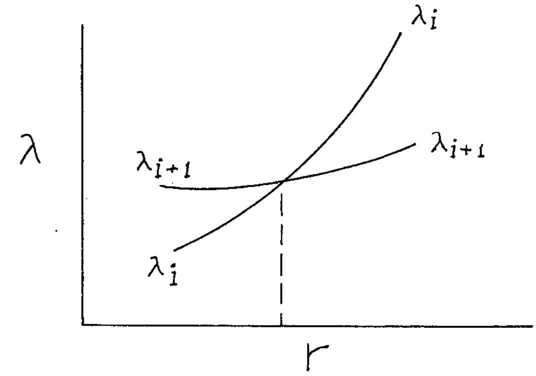

Fig. 5 Repeated frequencies lead to multiple partial derivatives.

It is hard to ascribe a physical interpretation to the assumptions contained in Eq. (23). However, for symmetrical systems, such as those described by Figs. 1–3, an infinitesimal parameter modification at the polar angle ϕ will cause a pair of repeated eigenvalue modes, $\{Z_i\}$ and $\{Z_{i+1}\}$, to lie in the ϕ and $\phi+(\pi/2)$ planes; their corresponding partial derivatives, $\{Z_i'\}$ and $\{Z_{i+1}'\}$, will also be contained in the same respective planes. Thus, both terms of Eq. (23) will be identically zero, as will the right-hand side of Eq. (22).

Equations (22) and (23) then combine to give

$$\{Z_k\}^T[B]\{Z_j'\} = -\tfrac{1}{2}\{Z_k\}^T[B']\{Z_j\} \qquad k, j = i, i+i, \ldots, i+m \tag{24}$$

and substitution of Eq. (19) into Eq. (24) yields

$$C_{jk} = b_{jk} - \{V_j\}^T[B]\{Z_k\}$$
$$b_{jk} = -\tfrac{1}{2}\{Z_k\}^T[B']\{Z_j\} \qquad j, k = i, i+1, \ldots, i+m \tag{25}$$

Thus, Eqs. (21) and (25) will be of identical form. It should be noted that although $b_{kj} = b_{jk}$, $C_{kj} \neq C_{jk}$ in Eq. (25).

Examples

Let us consider the simple two-degree-of-freedom system of Fig. 6 where the connecting rod of length L is massless along its length, but has end mass values m_1 and m_2. Selecting the generalized coordinates X_1 and $L\theta$, the system kinetic and potential energies T and V, respectively, are for small motions, as

$$T = \tfrac{1}{2}M_1\dot{X}_1^2 + \tfrac{1}{2}M_2(\dot{X}_1 - L\dot{\theta})^2 \tag{26}$$

and

$$V = \tfrac{1}{2}K_1X_1^2 + \tfrac{1}{2}K_2(X_1 - L\theta)^2 \tag{27}$$

Applicating Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0, \qquad q_i = X_1,\ q_2 = \theta \tag{28}$$

and assuming harmonic motion yields the 2 × 2 matrix eigenvalue equation

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} X_1 \\ L\theta \end{Bmatrix} = \lambda \begin{bmatrix} m_1+m_2 & -m_2 \\ -m_2 & m_2 \end{bmatrix}\begin{Bmatrix} X_1 \\ L\theta \end{Bmatrix} \tag{29}$$

for the natural frequencies squared (eigenvalues)

$$\lambda_1 = k_1/m_1, \qquad \lambda_2 = k_2/m_2 \tag{30}$$

For the case where $k_1 = k_2 = K$ and $m_1 = m_2 = M$, the eigenvalues of this problem, λ_1 and λ_2, coalesce and become simply

$$\lambda_1 = \lambda_2 = K/M \tag{31}$$

and the orthonormal mode shapes may be represented as

$$(2M)^{-\frac{1}{2}}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \qquad (2M)^{-\frac{1}{2}}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \tag{32}$$

A second example treated by the proposed repeated-root gradient theory involves a square, cantilevered, classical beam with additional tip masses m_a and m_b (see Fig. 7). When m_a equals m_b, the bending frequencies occur in equal pairs and any plane containing the x axis may completely contain all mode-shape activity. However, to obtain gradients with respect to changes in m_a or m_b only, the original modal vectors must be contained in the x-$\bar{y}$ and x-$\bar{z}$ planes. To verify if the proposed procedure achieves this, we shall start with initial mode shapes $\{X\}$ in the x-y and x-z planes and demonstrate that the new modes and their gradients $\{Z\}$ and $\{Z'\}$, respectively, end up in the x-$\bar{y}$ and x-$\bar{z}$ planes.

The stiffness and diagonal mass matrices for the two-element finite-element beam shown in Fig. 7, are

$$[K] = 100\begin{bmatrix} 12 & 0 & 0 & 0 & -6 & 3 & 0 & 0 \\ & 4 & 0 & 0 & -3 & 0 & 0 & 0 \\ & & 12 & 0 & 0 & 0 & -6 & 3 \\ & & & 4 & 0 & 0 & -3 & 1 \\ & & & & 6 & -3 & 0 & 0 \\ & \text{symmetric} & & & & 2 & 0 & 0 \\ & & & & & & 6 & -3 \\ & & & & & & & 2 \end{bmatrix}$$

$$[M] = \tfrac{1}{100}\begin{bmatrix} 2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0.208 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 2.5 & 0 & 0 & 0 & 0 & 0 \\ & & & 0.208 & 0 & 0 & 0 & 0 \\ & & & & 21.25 & 0 & 0 & 0 \\ & \text{symmetric} & & & & 10.1 & 0 & 0 \\ & & & & & & 21.25 & 0 \\ & & & & & & & 10.1 \end{bmatrix}$$

The degrees of freedom corresponding to eigenvectors $\{X\}$ are

$$\{X\}^T = \lfloor U_1, \theta_1, V_1, \phi_1, U_2, \theta_2, V_2, \phi_2 \rfloor$$

The first bending solutions $\{X^{(1)}\}$ and $\{X^{(2)}\}$ corresponding to $[K]\{X\} = \lambda[M]\{X\}$ are

$$\lambda_1 = \lambda_2 = 67.84$$

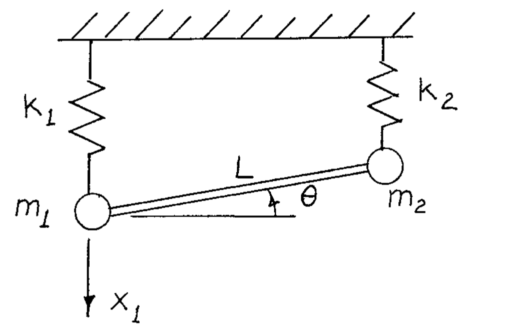

Fig. 6 Vibrations of a simple two-degree-of-freedom system.

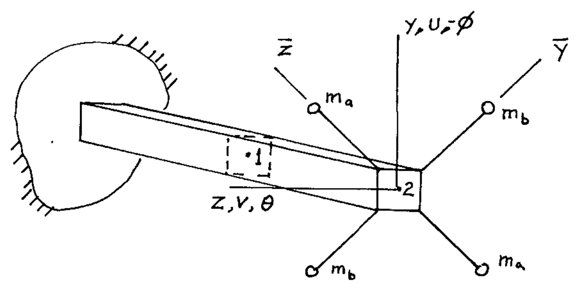

Fig. 7 Square cantilevered beam with offset end masses.

and

$$\{X^{(1)}\}^T = \lfloor 0,0,0.565,1.037,0,0,1.888,1.517 \rfloor$$

$$\{X^{(2)}\}^T = \lfloor 0.565,1.037,0,0,1.888,1.517,0,0 \rfloor$$

The eigenvalue gradients with respect to m_b for the problem require $\{K'\}$ and $[M']$, where $[K']$ is an 8×8 null matrix and

$$[M'] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Application of Eqs. (17) and (18) then yields

$$[D] = \begin{bmatrix} -639.6 & -156.1 \\ -156.1 & -639.6 \end{bmatrix}$$

$$\lambda_1' = -795.75, \qquad \lambda_2' = -483.49$$

$$\{\alpha_1\} = \begin{Bmatrix} 0.7071 \\ 0.7071 \end{Bmatrix}, \qquad \{\alpha_2\} = \begin{Bmatrix} 0.7071 \\ -0.7071 \end{Bmatrix}$$

while Eq. (12) yields

$$\{Z^{(1)}\} = \begin{Bmatrix} 0.400 \\ 0.733 \\ 0.400 \\ 0.733 \\ 1.335 \\ 1.073 \\ 1.335 \\ 1.073 \end{Bmatrix}, \qquad \{Z^{(2)}\} = \begin{Bmatrix} -0.400 \\ -0.733 \\ 0.400 \\ 0.733 \\ -1.335 \\ 1.072 \\ 1.335 \\ 1.073 \end{Bmatrix}$$

Physically speaking, mode 1 corresponds to both masses moving up or down simultaneously (together), while mode 2 corresponds to m_1 and m_2 moving in opposite directions an equal amount, simultaneously.

Defining $(\ )' = \partial(\ )/\partial m_2$, Eq. (17) becomes

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \frac{2M^2}{K} \lambda' \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \tag{33}$$

which has eigenvalue solutions

$$\lambda' = 0, -K/M^2 \tag{34}$$

and corresponding eigenvector solutions

$$\{\alpha^{(1)}\} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \qquad \{\alpha^{(2)}\} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \tag{35}$$

Thus,

$$\{Z^{(1)}\} = \alpha_1^{(1)} \begin{Bmatrix} X_1^{(1)} \\ L\theta^{(1)} \end{Bmatrix} + \alpha_2^{(1)} \begin{Bmatrix} X_1^{(2)} \\ L\theta^{(2)} \end{Bmatrix} = \frac{1}{\sqrt{m}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{36}$$

and

$$\{Z^{(2)}\} = \alpha_1^{(2)} \begin{Bmatrix} X_1^{(1)} \\ L\theta^{(1)} \end{Bmatrix} + \alpha_2^{(2)} \begin{Bmatrix} X_1^{(2)} \\ L\theta^{(2)} \end{Bmatrix} = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} \tag{37}$$

A physical interpretation of $\{Z^{(1)}\}$ yields mass 1 moving up and down while mass 2 is stationary; $\{Z^{(2)}\}$ corresponds to mass 1 remaining stationary while mass 2 moves vertically.

Equation (14) disappears for this trivial example, leaving $\{V_1\}$ and $\{V_2\}$ equal to zero. Equation (19) now becomes

$$\{Z_1'\} = C_{11}\{Z^{(1)}\} + C_{12}\{Z^{(2)}\}$$
$$\{Z_2'\} = C_{21}\{Z^{(1)}\} + C_{22}\{Z^{(2)}\} \tag{38}$$

and Eq. (25) yields

$$C_{11} = 0, \qquad C_{12} = C_{21} = 0$$
$$C_{22} = -1/(2M) \tag{39}$$

Thus,

$$\{Z_1'\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

and

$$\{Z_2'\} = \begin{Bmatrix} 0 \\ -(2M)^{-\frac{3}{2}} \end{Bmatrix} \tag{40}$$

A physical interpretation of this result is that increasing m_2 slightly will not perturb mode shape 1 at all, while it will affect mode shape 2 only by decreasing its amplitude such that its new normalization condition is still given by Eq. (3) or

$$\left(\{Z_2\}^T[B]\{Z_2\} - 1\right)' = 0 \tag{41}$$

Inspection of $\{Z^{(1)}\}$ and $\{Z^{(2)}\}$ gives p and q equal to 5 and 7. Thus, application of Eqs. (20), (25), and (19) yields

$$\{F_1\} = \begin{Bmatrix} -7.954 \\ -1.216 \\ -7.954 \\ -1.216 \\ 0 \\ 59.3 \\ 0 \\ 59.3 \end{Bmatrix}, \qquad \{F_2\} = \begin{Bmatrix} 4.833 \\ 0.739 \\ -4.833 \\ -0.739 \\ 0 \\ 52.4 \\ 0 \\ -52.4 \end{Bmatrix}$$

$$\{V_1\} = \begin{Bmatrix} -0.172 \\ -0.168 \\ -0.172 \\ -0.168 \\ 0 \\ 0.662 \\ 0 \\ 0.662 \end{Bmatrix}, \qquad \{V_2\} = \begin{Bmatrix} -0.133 \\ -0.135 \\ 0.133 \\ 0.135 \\ 0 \\ 0.548 \\ 0 \\ -0.548 \end{Bmatrix}$$

$$C_{11} = -0.6004, \qquad C_{12} = -0.1447$$
$$C_{21} = -0.1189, \qquad C_{22} = -3.448$$

$$\{Z_1'\} = \begin{Bmatrix} -2.525 \\ -4.466 \\ -2.630 \\ -4.676 \\ -7.823 \\ -5.625 \\ -8.206 \\ -5.934 \end{Bmatrix}, \qquad \{Z_2'\} = \begin{Bmatrix} 1.198 \\ 2.305 \\ -1.293 \\ -2.480 \\ 4.443 \\ 4.119 \\ -4.761 \\ -4.374 \end{Bmatrix}$$

This problem was also solved using a numerical perturbation and differencing scheme. The results for λ_1' and λ_2' were within 0.03% of the above values, while the results for $\{Z_1'\}$ and $\{Z_2'\}$ departed at most by 3% for any given element from those presented above.

Directional Derivatives

Up to this point, attention has been directed at a single parameter variation. However, eigenvector derivatives are important for structural design and optimization when many parameters are varied simultaneously. Under such circumstances, it is important to proceed with extreme caution since the associated eigenvector changes may not be continuous and, therefore, nondifferentiable. To explain this physically, let us once again consider the system of Fig. 1. If the stiffness of two springs are varied independently, then the eigenvectors associated with each change will be different and discontinuous if the parameters are modified simultaneously and independently. Thus, we may consider design changes only when one parameter is changed at a time or if we select an a priori fixed ratio between the design changes. We shall now discuss this in more mathematical terms.

Consider an eigenvector $\{Z^{(i)}\}$ that depends upon an arbitrary set of parameters, r_j, $j=1,2,\ldots,S$. If the eigenvector varies smoothly for our arbitrary set of parameter changes δr_j, then it follows that

$$\delta\{Z^{(i)}\}=\sum_{j=1}^{S}\frac{\partial\{Z^{(i)}\}}{\partial r_j}\delta r_j \tag{42}$$

where S is the number of variables r_j and $\delta\{Z^{(i)}\}$ an infinitesimally small increment of the eigenvector.

Equation (42) will not generally be true in the presence of repeated eigenvalues. This fact is related to the ambiguity of the eigenvectors when repeated eigenvalues occur. Thus, even though the partial derivatives

$$\frac{\partial Z^{(i)}}{\partial r_j}$$

can be calculated according to the procedures outlined earlier, Eq. (42) will not always hold. There are special instances, however, when Eq. (42) is valid. These instances are related to particular directional increments δF, where

$$\delta F=\sum_j a_j\,\delta r_j \tag{43}$$

and Eq. (42) becomes

$$\delta\{Z^{(i)}\}=\frac{\partial\{Z^{(i)}\}}{\partial F}\delta F \tag{44}$$

in which case

$$\frac{\partial\{z^{(i)}\}}{\partial F}$$

is called the directional derivative and the a_j are given by

$$a_j=\frac{\partial F}{\partial r_j} \tag{45}$$

Then, Eq. (43) has physical validity.

Thus, if all the changes δr_j simultaneously refer to the same original eigenvector Z_i, then all such variations may be considered simultaneously. An example of this for Fig. 2 is to vary the shell radius and wall thickness as a function of shell length, but not polar angle.

Hence, determination of the directions in which Eqs. (43–45) are valid will generally require some physical insight and user input.

Discussion

The previous examples are an interesting and simple demonstration of the insight that may be gained by following the proposed procedure. The coordinate system and mode shapes initially selected for the examples gave little clue to how each mode would respond under change. However, when we proposed perturbing only one of the mass parameters, the proper modes emerged along with the correct partial derivatives.

Thus, it is seen that the proposed mathematical approach can automatically yield physical insight regarding modal sensitivities to parameter modifications without ambiguity or user dependence. Moreover, the uncertainties and indeterminancies associated with repeated roots may be removed through investigation of specific parameter modifications.

It is important to note that the present approach may be used equally well for closely spaced modes. Such modes are sometimes associated with poorly conditioned matrices and may depend upon machine precision for their separation. However, the present method permits some insight by which these modes can be separated, regardless of how close they may be, through consideration of their partial derivatives.

We have also discussed the caution necessary when working with repeated eigenvalue problems, in light of the work contained in Refs. 4–6 and the existence of only directional derivatives.

The present work has built upon the computationally efficient ideas of Nelson[1] and the repeated eigenvalue derivative equation of Haug and his co-workers[4–6] and Chen and Pan.[2] We have extended Nelson's computational efficiency through proper elimination of additional redundancies in the system matrices while maintaining the original matrix bandwidths. In addition, the newly proposed procedure is more direct than the modal expansion techniques proposed in Ref. 2, is more suitable for large problems with many degrees of freedom, and introduces the need for consideration of the modal orthogonality gradient equations.

As noted earlier, the C_{jk} of Eq. (19) are not uniquely determined if $j\neq k$. Other choices are possible, since Eq. (23) is arbitrary and not necessary for their determination. We do not know if this can result in eigenvector derivative ambiguity or error. It is believed that further work in this area, as well as efforts to study the effects when eigenvalue sensitivities themselves repeat, is also needed. The application of directional derivatives for optimization studies in the presence of repeated eigenvalue also remains an unresolved issue and additional work is needed here as well.

Although the present work has focused upon symmetric banded matrices, we see no conceptional reasons why the above procedures cannot be systematically expanded to include complex, unsymmetric matrices with complex eigenvalues and biorthogonal eigenvectors.

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